Math 429 - Exercise Sheet 13

1. Prove Proposition 32.

2. Prove Proposition 34.

3. Prove that for any $\lambda, \mu \in \mathfrak{h}^*$, there exists an injective $\mathfrak{g}$-intertwiner

$$L(\lambda + \mu) \hookrightarrow L(\lambda) \otimes L(\mu)$$

(where $L(\lambda)$ denotes the irreducible representation of $\mathfrak{g}$ with highest weight λ, as in the notes).

4. Work out the weight decomposition of the symmetric power representation $V = S^k\mathbb{C}^n$ of $\mathfrak{sl}_n$, i.e. write down those $\lambda \in P$ for which $V_\lambda \neq 0$.

5. Consider the tautological representation of $\mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$ respectively (namely $\mathbb{C}^{2n+1}, \mathbb{C}^{2n}, \mathbb{C}^{2n}$ respectively). Determine the highest weight of these three representations; recall our conventions on the root systems of types B, C, D and their simple roots in Ex. 1, Sheet 10 and Ex. 1, Sheet 11.

(*) Let V be a representation of a finite-dimensional abelian Lie algebra $\mathfrak{h}$, which admits a weight decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \qquad \text{where } V_\lambda = \{v \in V | x \cdot v = \lambda(x)v, \forall x \in \mathfrak{h}\}$$

If the weight spaces V_λ are all finite-dimensional, then show that any subrepresentation and quotient representation of V also admits a weight decomposition.